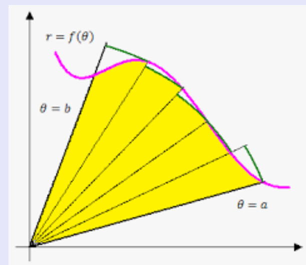
Calculus II

Lecture 10

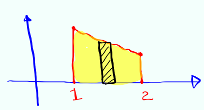


Feb 19-8:47 AM

Find the area below $f(x) = \frac{1}{x^3+x}$, above x-axis, from $x=1$ to $x=2$. Drawing required.

$f(1) = \frac{1}{2}$, $f(2) = \frac{1}{10}$

$f(x) \geq 0$ on $[1,2]$



$$A = \int_1^2 \frac{1}{x^3+x}\,dx$$

$$\frac{1}{x^3+x} = \frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$1 = A(x^2+1) + (Bx+C)x$$

$$1 = Ax^2 + A + Bx^2 + Cx$$

$$1 = (A+B)x^2 + Cx + A$$

$A + B = 0 \rightarrow B = -1$

$C = 0$

$A = 1$

$$\int_1^2 \frac{1}{x^3+x}\,dx =$$

$$\int_1^2 \left(\frac{1}{x} + \frac{-x+0}{x^2+1}\right)dx$$

$$= \int_1^2 \left[\frac{1}{x} - \frac{x}{x^2+1}\right]dx$$

$\int \frac{1}{x}\,dx = \ln x$

$\int \frac{x}{x^2+1}\,dx$

$u = x^2+1$

$du = 2x\,dx$

$\frac{du}{2} = x\,dx$

$$\int \frac{x}{x^2+1}\,dx = \int \frac{1}{u}\,\frac{du}{2}$$

$$= \frac{1}{2}\ln u = \frac{1}{2}\ln(x^2+1)$$

$$= \left[\ln x - \frac{1}{2}\ln(x^2+1)\right]\Big|_1^2$$

$$= \ln \frac{x}{\sqrt{x^2+1}}\Big|_1^2$$

$$= \ln \frac{2}{\sqrt{5}} - \ln \frac{1}{\sqrt{2}} = \ln \frac{\frac{2}{\sqrt{5}}}{\frac{1}{\sqrt{2}}}$$

$$= \ln \frac{2\sqrt{2}}{\sqrt{5}} = \boxed{\ln \frac{2\sqrt{10}}{5}}$$

Jun 25-11:00 AM

$$\int \frac{x^2+2x-1}{x^3-x}\,dx$$

Doing Partial Fractions

$$\frac{x^2+2x-1}{x^3-x} = \frac{x^2+2x-1}{x(x^2-1)} = \frac{x^2+2x-1}{x(x+1)(x-1)} = \frac{A}{x}+\frac{B}{x+1}+\frac{C}{x-1}$$

$$x^2+2x-1 = A(x+1)(x-1) + B\,x(x-1) + C\,x(x+1)$$

$x=0$ $\quad -1 = A(1)(-1) + B\cdot 0 + C\cdot 0$ $\quad \boxed{A=1}$

$x=1$ $\quad 2 = A\cdot 0 + B\cdot 0 + C\cdot 2$ $\quad \boxed{C=1}$

$x=-1$ $\quad -2 = A\cdot 0 + B\cdot 2 + C\cdot 0$ $\quad \boxed{B=-1}$

$$\int \frac{x^2+2x-1}{x^3-x}\,dx = \int\left[\frac{1}{x} - \frac{1}{x+1} + \frac{1}{x-1}\right]dx$$

$$= \ln x - \ln(x+1) + \ln(x-1) + C$$

$$= \boxed{\ln\left|\frac{x(x-1)}{x+1}\right| + C}$$

Jun 26-8:14 AM

$$\int \frac{x^2+x+1}{(x^2+1)^2}\,dx$$

Doing Partial Fractions

$$\frac{x^2+x+1}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}$$

$$x^2+x+1 = (Ax+B)(x^2+1) + Cx + D$$

$$= Ax^3 + Ax + Bx^2 + B + Cx + D$$

$$= Ax^3 + Bx^2 + (A+C)x + B + D$$

No x^3

$0x^3$

$\boxed{A=0}$

$\boxed{1=B}$

$A+C=1$ $\quad$ $B+D=1$

$0+C=1$ $\quad$ $1+D=1$

$\boxed{C=1}$ $\quad$ $\boxed{D=0}$

$$\int \frac{x^2+x+1}{(x^2+1)^2}\,dx = \int\left[\frac{0x+1}{x^2+1} + \frac{1x+0}{(x^2+1)^2}\right]dx$$

Make Sure to Finish

$$= \int \frac{1}{x^2+1}\,dx + \int \frac{x}{(x^2+1)^2}\,dx = \tan^{-1}x + \int \frac{1}{u^2}\frac{du}{2}$$

$u = x^2+1$

$du = 2x\,dx \rightarrow \frac{du}{2} = x\,dx$

Jun 26-8:24 AM

$$\int \frac{x^5+x-1}{x^3+1}\,dx$$

Since the degree of the numerator is not less than the degree of the denominator, then we must do long division

$$\begin{array}{r} x^2 \\ x^3+1\,\overline{)\,x^5+0x^4+0x^3+0x^2+x-1} \\ -(x^5 \qquad\qquad +x^2) \\ \hline -x^2+x-1 \end{array}$$

$$\int \frac{x^5+x-1}{x^3+1}\,dx = \int\left(x^2 + \frac{-x^2+x-1}{x^3+1}\right)dx$$

$$= \frac{x^3}{3} - \int \frac{x^2-x+1}{x^3+1}\,dx$$

Partial fractions

$$= \frac{x^3}{3} - \int \frac{1}{x+1}\,dx$$

$$\frac{x^2-x+1}{x^3+1} = \frac{\cancel{x^2-x+1}}{(x+1)(\cancel{x^2-x+1})} = \frac{1}{x+1}$$

$$= \frac{x^3}{3} - \ln|x+1| + C$$

$$A^3+B^3 = (A+B)(A^2-AB+B^2)$$

Jun 26-8:36 AM

$$\int_0^1 \frac{1}{1+\sqrt[3]{x}}\,dx$$

$u = \sqrt[3]{x}$ $\qquad x=0 \to u=0$

$u^3 = x$ $\qquad x=1 \to u=1$

$3u^2\,du = dx$

$$= \int_0^1 \frac{3u^2}{1+u}\,du = 3\int_0^1 \frac{u^2}{1+u}\,du$$

$$\begin{array}{r} u-1 \\ 1+u\,\overline{)\,u^2+0u+0} \\ -(u^2+u\quad) \\ \hline -u+0 \\ -(-u-1) \\ \hline 1 \end{array}$$

$$= 3\int_0^1 \left[u-1+\frac{1}{1+u}\right]du$$

$$= 3\left[\frac{u^2}{2} - u + \ln(1+u)\right]\Big|_0^1 = 3\left[\frac{1}{2} - 1 + \ln 2 - 0\right]$$

$$= \boxed{3\left[\ln 2 - \frac{1}{2}\right]}$$

Jun 26-8:46 AM

$$\int \frac{1}{\sqrt{x}-\sqrt[3]{x}}\,dx$$

Hint : $u=\sqrt[6]{x}$

$u^6 = x$

$6u^5\,du = dx$

$\sqrt{x} = \sqrt{u^6} = u^3$

$\sqrt[3]{x} = \sqrt[3]{u^6} = u^2$

$$= \int \frac{6u^5}{u^3-u^2}\,du$$

$$= 6\int \frac{u^5}{u^2(u-1)}\,du = 6\int \frac{u^3}{u-1}\,du$$

$$\begin{array}{r|l} & u^2 + u + 1 \\ \hline u-1 & u^3 + 0u^2 + 0u + 0 \\ & -(u^3 - u^2) \\ \hline & u^2 + 0u + 0 \\ & -(u^2 - u) \\ \hline & u + 0 \\ & -(u-1) \\ \hline & 1 \end{array}$$

$$= 6\int \left[u^2 + u + 1 + \frac{1}{u-1}\right] du$$

$$= 6\left[\frac{u^3}{3} + \frac{u^2}{2} + u + \ln|u-1|\right] + C$$

$$= 2\left(\sqrt[6]{x}\right)^3 + 3\left(\sqrt[6]{x}\right)^2 + 6\sqrt[6]{x}$$

$$+ 6\ln\left|\sqrt[6]{x}-1\right| + C$$

Jun 26-8:52 AM

$$\int \frac{e^x}{(e^x-2)(e^{2x}+1)}\,dx$$

Let $u = e^x$

$du = e^x\,dx$

$e^{2x} = (e^x)^2 = u^2$

$$= \int \frac{1}{(u-2)(u^2+1)}\,du$$

$$\frac{1}{(u-2)(u^2+1)} = \frac{A}{u-2} + \frac{Bu+C}{u^2+1}$$

$$1 = A(u^2+1) + (Bu+C)(u-2)$$

$u=2$ $\quad 1 = A\cdot 5 \rightarrow A = \frac{1}{5}$

$u=0$ $\quad 1 = A\cdot 1 + (\cancel{B\cdot 0} + C)(-2)$

$1 = \frac{1}{5} - 2C \qquad 5 = 1 - 10C$

$10C = -4$

$\boxed{C = \frac{-2}{5}}$

$u=1$ $\quad 1 = \frac{1}{5}\cdot 2 + \left(B - \frac{2}{5}\right)(-1)$

$1 = \frac{2}{5} - B + \frac{2}{5} \qquad 5 = 2 - 5B + 2$

$5B = -1$

$$\int \frac{1}{(u-2)(u^2+1)}\,du = \int \frac{\frac{1}{5}}{u-2}\,du + \int \frac{-\frac{1}{5}u - \frac{2}{5}}{u^2+1}\,du \qquad \boxed{B = \frac{-1}{5}}$$

Jun 26-9:05 AM

$$\int \frac{1}{(u-2)(u^2+1)}\,du = \int \frac{\frac{1}{5}}{u-2}\,du + \int \frac{-\frac{1}{5}u - \frac{2}{5}}{u^2+1}\,du$$

$$= \frac{1}{5}\int \frac{1}{u-2}\,du - \frac{1}{5}\int \frac{u+2}{u^2+1}\,du$$

$$= \frac{1}{5}\left\{ \int \frac{1}{u-2}\,du - \int \frac{u}{u^2+1}\,du - \int \frac{2}{u^2+1}\,du \right\}$$

$w = u^2+1$

$dw = 2u\,du$

$$= \frac{1}{5}\left\{ \ln|u-2| - \frac{1}{2}\ln(u^2+1) - 2\tan^{-1}u \right\} + C$$

$$= \frac{1}{5}\left\{ \ln|e^x-2| - \frac{1}{2}\ln(e^{2x}+1) - 2\tan^{-1}e^x \right\} + C$$

Jun 26-9:16 AM

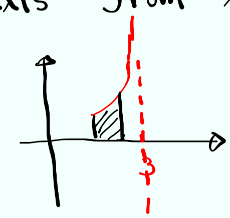
Find the area below $f(x) = \dfrac{x^2+1}{3x-x^2}$, above x-axis from $x=1$ to $x=2$.

$3x - x^2 = 0$

$x(3-x) = 0$

$x=0 \quad x=3$

V.A.



$$A = \int_1^2 \frac{x^2+1}{3x-x^2}\,dx$$

$$= \int_1^2 \left[-1 + \frac{3x+1}{3x-x^2}\right]dx$$

$$\begin{array}{r} -1 \\ -x^2+3x \,\big)\, x^2 + 0x + 1 \\ -(x^2 - 3x) \\ \hline 3x + 1 \end{array}$$

$$= \int_1^2 \left[-1 + \frac{\frac{1}{3}}{x} + \frac{\frac{10}{3}}{3-x}\right]dx$$

$$\frac{3x+1}{3x-x^2} = \frac{3x+1}{x(3-x)} = \frac{A}{x} + \frac{B}{3-x}$$

$$= \left[-x + \frac{1}{3}\ln x - \frac{10}{3}\ln(x-3)\right]_1^2$$

$3x + 1 = A(3-x) + Bx$

$x=0 \quad 1 = 3A \quad \rightarrow A = \frac{1}{3}$

$x=3 \quad 10 = 3B \quad \rightarrow B = \frac{10}{3}$

$=$

Jun 26-9:22 AM

Arc length of $y=f(x)$ on $[a,b]$

$$L=\int_a^b \sqrt{1+[f'(x)]^2}\,dx$$

$$f(x)=\frac{x^3}{3}+\frac{1}{4x}\ ,\ [1,2]$$

$$f'(x)=\frac{3x^2}{3}-\frac{1}{4x^2} \qquad f'(x)=x^2-\frac{1}{4x^2}$$

$$1+[f'(x)]^2=1+\left[x^2-\frac{1}{4x^2}\right]^2$$

$$=1+x^4-2\cdot x^2\cdot\frac{1}{4x^2}+\frac{1}{16x^4}$$

$$=1+x^4-\frac{1}{2}+\frac{1}{16x^4}$$

$$=x^4+\frac{1}{2}+\frac{1}{16x^4}=\left(x^2+\frac{1}{4x^2}\right)^2$$

$$L=\int_1^2\sqrt{\left(x^2+\frac{1}{4x^2}\right)^2}\,dx=\int_1^2\left(x^2+\frac{1}{4x^2}\right)dx$$

$$=\left(\frac{x^3}{3}+\frac{1}{4}\cdot\frac{x^{-1}}{-1}\right)\Big|_1^2=$$

Jun 26-10:01 AM

Arc length of $x=f(y)$ on $[c,d]$

$$L=\int_c^d\sqrt{1+[f'(y)]^2}\,dy$$

$$x=\frac{1}{3}\sqrt{y}\,(y-3)\ ,\ [1,9]$$

$$x=\frac{1}{3}y^{\frac{3}{2}}-y^{1/2} \qquad f(y)=\frac{1}{3}y^{3/2}-y^{1/2}$$

$$f'(y)=\frac{1}{\cancel{3}}\cdot\frac{\cancel{3}}{2}\cdot y^{1/2}-\frac{1}{2}y^{-1/2}$$

$$1+[f'(y)]^2=1+\frac{1}{4}\left(\sqrt{y}-\frac{1}{\sqrt{y}}\right)^2 \qquad =\frac{1}{2}\left[\sqrt{y}-\frac{1}{\sqrt{y}}\right]$$

$$=1+\frac{1}{4}\left[y-2\sqrt{y}\cdot\frac{1}{\sqrt{y}}+\frac{1}{y}\right]$$

$$=1+\frac{1}{4}\left[y-2+\frac{1}{y}\right]$$

$$=\frac{1}{4}y-\frac{1}{2}+\frac{1}{4}\cdot\frac{1}{y}+1$$

$$=\frac{1}{4}y+\frac{1}{2}+\frac{1}{4y}=\left(\frac{1}{2}\sqrt{y}+\frac{1}{2\sqrt{y}}\right)^2$$

$$\cancel{2}\cdot\frac{1}{\cancel{2}}\cancel{\sqrt{y}}\cdot\frac{1}{2\cancel{\sqrt{y}}} \qquad (A+B)^2=A^2+2AB+B^2$$

$$L=\int_1^9\sqrt{\left(\frac{1}{2}\sqrt{y}+\frac{1}{2\sqrt{y}}\right)^2}\,dy=\int_1^9\left[\frac{1}{2}y^{1/2}+\frac{1}{2}y^{-1/2}\right]dy$$

$$=\frac{1}{2}\left[\frac{y^{3/2}}{3/2}+\frac{y^{1/2}}{1/2}\right]\Big|_1^9=$$

Jun 26-10:08 AM

$y = \ln(\cos x) \qquad 0 \le x \le \frac{\pi}{3}$

$f(x) = \ln(\cos x)$

$f'(x) = \dfrac{-\sin x}{\cos x}$

$$L = \int_0^{\pi/3} \sqrt{1 + \left[\frac{-\sin x}{\cos x}\right]^2}\, dx$$

$$= \int_0^{\pi/3} \sqrt{1 + \frac{\sin^2 x}{\cos^2 x}}\, dx = \int_0^{\pi/3} \sqrt{\frac{\sin^2 x + \cos^2 x}{\cos^2 x}}\, dx$$

$$= \int_0^{\pi/3} \frac{1}{\cos x}\, dx = \int_0^{\pi/3} \sec x\, dx = \ln|\sec x + \tan x| \Big|_0^{\pi/3}$$

$=$

Jun 26-10:21 AM

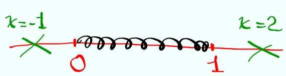
Find the arc length of

$f(x) = \sqrt{x - x^2} + \sin^{-1}\sqrt{x}. \qquad [0,1]$

$x \ge 0$

$x - x^2 \ge 0$

$x(1 - x) \ge 0$



$f(x) = (x - x^2)^{1/2} + \sin^{-1}(\sqrt{x})$

$f'(x) = \frac{1}{2}(x - x^2)^{-1/2} \cdot (1 - 2x) + \dfrac{1}{\sqrt{1 - (\sqrt{x})^2}} \cdot \dfrac{1}{2\sqrt{x}}$

$f'(x) = \dfrac{1 - 2x}{2\sqrt{x - x^2}} + \dfrac{1}{2\sqrt{x}\sqrt{1 - x}} = \dfrac{2 - 2x}{2\sqrt{x - x^2}}$

$f'(x) = \dfrac{1 - x}{\sqrt{x - x^2}}, \quad 1 + [f'(x)]^2 = 1 + \dfrac{1 - 2x + x^2}{x - x^2}$

$= \dfrac{x - \cancel{x^2} + 1 - 2x + \cancel{x^2}}{x - x^2}$

$= \dfrac{1 - x}{x - x^2} = \dfrac{1 - x}{x(1 - x)} = \dfrac{1}{x}$

$L = \int_0^1 \sqrt{1 + [f'(x)]^2}\, dx$

$= \int_0^1 \sqrt{\frac{1}{x}}\, dx = \int_0^1 x^{-1/2}\, dx$

$= \dfrac{x^{1/2}}{1/2}\Big|_0^1 = 2\sqrt{x}\Big|_0^1 = \boxed{2}$

Jun 26-10:27 AM

Surface Area when $y=f(x)$ on $[a,b]$ is rotated about x-axis

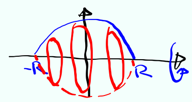
$$S=\int_a^b 2\pi f(x)\sqrt{1+[f'(x)]^2}\,dx$$

$\underbrace{f(x)=\sqrt{R^2-x^2}}$, $[-R,R]$, find Surface Area.

$y=\sqrt{R^2-x^2}$

$y^2=R^2-x^2$

$x^2+y^2=R^2$



Sphere

$f(x)=\sqrt{R^2-x^2}$ $\quad f(x)=(R^2-x^2)^{1/2}$ $\quad f'(x)=\dfrac{-x}{\sqrt{R^2-x^2}}$

$$1+[f'(x)]^2=1+\frac{x^2}{R^2-x^2}=\frac{R^2}{R^2-x^2}$$

$$S=\int_{-R}^{R}2\pi\,\cancel{\sqrt{R^2-x^2}}\cdot\sqrt{\frac{R^2}{\cancel{R^2-x^2}}}\,dx=\int_{-R}^{R}2\pi R\,dx$$

$$=2\pi R\cdot x\Big|_{-R}^{R}=2\pi R(R-(-R))=\boxed{4\pi R^2}$$

Volume of sphere

$V=\dfrac{4\pi R^3}{3}$

Surface Area

$S=4\pi R^2$

Area of Circle

$A=\pi R^2$

Circumference of Circle

$C=2\pi R$

Jun 26-10:50 AM